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一类超静定变截面压杆临界荷载的优化算法

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摘　要 目的 研究一端固定,一端链杆约束的一类超静定变截面梁临界载荷的优化算法原理,探讨超静定压杆临界载荷数值方法. **方法** 基于差分原理和优化基本原理,运用差分方法将平衡状态下非线性微分方程离散化. 以杆件每个离散点挠度,临界荷载和多余约束力为设计变量,以临界载荷所满足差分方程与边界条件构建目标函数,在 Fortran-PowerStation 环境下,编制优化程序,进行常截面梁与变截面梁具体算例分析对比. **结果** 提出了一类超静定变截面压杆的临界载荷的无约束优化算法,并验证了方法的正确性和有效性. **结论** 算法能够有效解决工程中变截面超静定梁结构临界载荷的计算问题,为工程设计与分析提供支持.

关键词 超静定变截面压杆;临界载荷;非线性差分方程;优化算法;程序设计

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An Optimal Algorithm for the Critical Load of a Class of Statically Indeterminate Variable Cross-section Compression Bar

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Abstract: The principle of the optimization algorithm for the critical load of a statically indeterminate variable cross-section beam with one end fixed and one end chain rod constraint was studied, and the numerical method for the critical load of the static pressure bar was discussed. Based on finite difference method and optimization method, the nonlinear differential equations in equilibrium state were discretized. The bar deflections at each discrete point, the critical load and the extra binding force were took as the design variables. Critical load difference equation and boundary conditions were used to construct the objective function. An optimization algorithm procedure was built with the computer language Fortran-PowerStation and was verified with concrete examples of uniform bar and variable cross-section bar. optimization algorithm of the statically indeterminate variable cross-section beam was built, whose correctness and effectiveness. was verified The algorithm can solve the critical load calculating issue of statically indeterminate variable cross-section

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beam effectively and could provide some support to the engineering design and analysis.
Key words:statically indeterminate variable cross compression bar;critical load;nonlinear difference equations;optimization algorithm;program design

轴心受压承压杆件稳定临界载荷的计算是工程结构中重要问题. 根据压杆两端不同约束情况,形成静定约束承压杆和超静定约束承压杆,超静定约束承压杆未知量除临界载荷,还有多余的约束力. 若截面为常截面,运用材料力学或结构力学理论,直接通过欧拉公式和近似公式计算大柔度和中柔度压杆临界载荷[1-2]. 而对于变截面压杆,通常没有相应临界载荷解析表达式,只能求解近似解. 多采用实验法、传递函数和有限单元法等方法,理论分析与计算过程都较为繁琐[3-9].

变截面杆件稳定性问题分析实质是求解含有临界载荷的非线性微分方程,构造有效的数值算法. 笔者等[10-11]曾针对简支梁静定变截面压杆临界载荷计算问题,通过差分法和优化算法的结合,提出了临界载荷的数值迭代算法和优化算法. 精确获得临界载荷的计算. 在此基础上,笔者引入动态设计变量优化原理[12-13],探讨一端固定,一端链杆约束的一类超静定变截面压杆稳定的临界载荷的优化计算方法,算法能够有效解决工程中变截面超静定梁结构临界载荷的计算问题,为工程设计与分析提供支持.

1　超静定变截面压杆临界载荷求解问题分析

图 1 为一端固定,一端链杆约束的超静定变截面受压杆达到一阶稳定状态的受力图. 图中压杆长度为 l,弯曲刚度为 $EI(x)$ 随坐标 x 变化. F_{cr}、F_B 分别为未知的临界载荷和多余约束力. 将压杆沿 x 坐标轴离散分成 n 段,$\Delta x = l/n$,节点的坐标:$x_i = i\Delta x$,$i = 0, 1, 2, \cdots, n$. w 为挠度值,对应挠度坐标为 w_i,$i = 0, 1, 2, \cdots, n$,挠曲轴线离散点坐标为 (x_i, w_i),$i = 0, 1, 2, \cdots, n$. 左端 A,$w_0 = w_0' = 0$,右端点 B,$w_n = 0$.

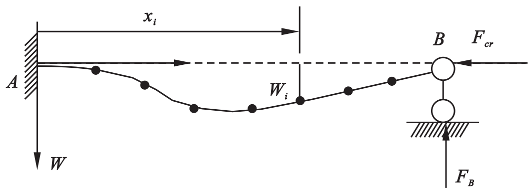

图 1　轴心受压杆件受力图

Fig. 1　Force diagram of the axis pressed rod

杆件挠曲轴线微分方程:

$$EI(x)w'' = M(x). \tag{1}$$

而 $M(x) = -F_{cr}w - F_B(l - x)$,将 $M(x)$ 代入式(1)得:

$$EI(x)w'' = -F_{cr}w - F_B(l - x). \tag{2}$$

应用差分原理,将导数项表示为差商公式 $w_i'' = \dfrac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2}$,代入变截面条件 $EI(x_i) = EI_i$,杆件离散化的充分方程为

$$EI_i \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2} + F_{cr} w_i + F_B (l - x_i) = 0, i = 1, 2, \cdots, n-1,$$

或:

$$\frac{EI_i}{EI_0}(w_{i+1} - 2w_i + w_{i-1}) + \frac{F_{cr}\Delta x^2}{EI_0} w_i + \frac{F_B \Delta x^2}{EI_0}(l - x_i) = 0,\ i = 1, 2, \cdots, n-1. \tag{3}$$

式中:$EI_0 = EI(0)$.

式(3)为含 w_i,$i = 1, 2, \cdots, n-1$、临界荷载 F_{cr} 和多余约束 F_B 未知量总数为 $n+1$,方程总数为 $n-1$ 的非线性差分方程.

考虑图 1 杆件左边界 A 点处为固定约束条件,该位置杆件无转角,通过补点可得:

$$2w_1 + \left(\frac{F_B \Delta x^2}{EI_0}\right) l = 0. \tag{4}$$

在临界力 F_{cr} 作用下,挠曲轴线构成一种形态,而挠度值 w 的大小具有相对关系的,即所求解为一组模态解. 设给定其中 K 点处 x_K 挠度值 w_K,为求解方便,增加方程:

$$w_K = c. \tag{5}$$

式中：c 为给定的正常数.

此时方程组中添加式(4)与式(5)，方程总个数为 $n+1$，含有 $w_i, i=1,2,\cdots,n-1, i\neq K$、临界力 F_{cr} 和多与约束 F_B 共 $n+1$ 个未知量，即挠曲轴线是唯一的，理论上可由式(3)唯一确定.

为了简化计算，若设：$\dfrac{EI_i}{EI_0}=b_i$，$\dfrac{w_i}{l}=W_i$，$\dfrac{F_{cr}\Delta x^2}{EI_0}=f_r$，$\dfrac{F_B\Delta x^2}{EI_0}=f_b$，代入式(3)，可得无量纲化差分方程：

$$b_i(W_{i+1}-2W_i+W_{i-1})+f_rW_i+f_b(1-W_i)=0, i=1,2,\cdots,n-1. \tag{6}$$

同样,对应的式(4),(5)的无量纲化方程为

$$2W_1+f_b=0, \tag{7}$$

$$W_K-c=0. \tag{8}$$

针对变截面问题形成的非线性差分方程组不能运用解析法求解，因此要构造临界荷载的优化求解思想[14-15].

2 超静定变截面压杆临界载荷最优化求解原理

2.1 临界载荷求解的最优化

超静定变截面压杆的临界荷载的无约束优化.

$$\min(f(z)). \tag{9}$$

式中：$z\in R^N$ 称为动态设计变量，$N=n+1$ 为动态设计变量 z 维数.

$$z_j=W_j, j=1,2\cdots,n-1, z_n=f_r, z_N=F_B.$$

$f(z)$ 为目标函数：

$$f(z)=\sum_{i=1}^{n-1}[(b_i(W_{i+1}-2W_i+W_{i-1})+f_rW_i+f_b(1-W_i)]^2+(W_K-c)^2+(2W_1+f_b)^2. \tag{10}$$

此优化形式的目的就是要通过对 $z=z^*$ 的搜寻得出所求优化函数的极小值 $f(z^*)\to 0$.

2.2 程序结构和流程

超静定刚架临界荷载的求解是将求解非线性方程组的过程转化为求解无约束优化的过程[16]，求解程序采用 Fortran - Powerstation 语言程序设计，程序组成包括：① 主程序；② 多维 Powell 法子程序；③进退法子程序；④ 黄金分割法子程序；⑤ 目标函数子段. 优化程序流程如图 2 所示.

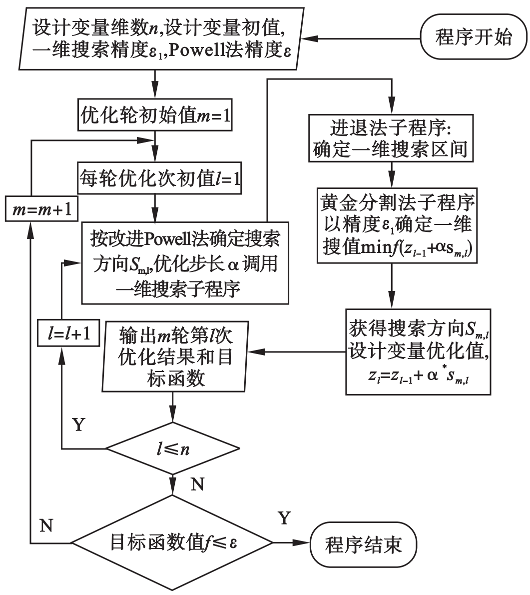

图 2 优化程序流程图

Fig. 2 Optimization program flowchart

2.3 计算精度判定条件

通过前后节点加密，所对应计算临界载荷值相对误差满足[17]：

$$\left|\frac{F_{cr}^{(k+1)}-F_{cr}^{(k)}}{F_{cr}^{(k)}}\right|<\varepsilon_r. \tag{11}$$

式中：$k=1,2,\cdots,n$.

节点位移相对误差满足[18]：

$$\sum_{i=1}^{n}\frac{1}{n}\left|\frac{w^{(k+1)}-w^{(k)}}{w^{(k)}}\right|<\varepsilon_r. \tag{12}$$

3 算例分析

3.1 常截面梁求解算例

常截面梁临界荷载的求解如图 1 所示梁杆件，杆件的长度为 l，杆件的截面弯曲刚度 EI，取设计变量数量 $n=21$ 时，取初值：$z_i=\mathrm{rand}()$，$i=1,2,\cdots,21$，其中，$z_1\sim z_{19}$ 表示挠度，z_{20} 表示无量纲临界荷载 f_r，z_{21} 表示无量纲

多余约束 f_b,多维 Powell 无约束精度 $e = 10^{-7}$,黄金分割精度 $e_1 = 10^{-4}$.

程序计算 8 568 轮,共 668 304 次,目标函数达到预设精度. 21 个变量计算结果见表 1.

表 1　$n = 21$ 设计变量的优化计算结果

Table 1　The optimization calculation results of design variables when $n = 21$

动态变量	计算结果	动态变量	计算结果	动态变量	计算结果
z_1	0. 019 3	z_8	0. 801 7	z_{15}	0. 903 2
z_2	0. 074 3	z_9	0. 913 2	z_{16}	0. 770 9
z_3	0. 160 3	z_{10}	1. 000 0	z_{17}	0. 607 5
z_4	0. 271 0	z_{11}	1. 056 0	z_{18}	0. 419 4
z_5	0. 399 1	z_{12}	1. 076 2	z_{19}	0. 214 1
z_6	0. 536 1	z_{13}	1. 057 9	z_{20}	0. 050 2
z_7	0. 673 2	z_{14}	0. 999 9	z_{21}	0. 038 6

对应的无量纲临界载荷和多余约束为

$$f_r = z_{20} = 0.050\,2, f_b = z_{21} = 0.038\,6.$$

对应的临界荷载与多余约束为

$$F_{cr} = \frac{f_r EI}{\Delta x^2} = 0.050\,2 \times 20^2 \frac{EI}{l^2} = 20.08 \frac{EI}{l^2},$$

$$F_B = \frac{f_b EI}{\Delta x^2} = 0.038\,6 \times 20^2 \frac{EI}{l^2} = 15.44 \frac{EI}{l^2}.$$

将该结果与准确值对比:

$$\left| \frac{F_{cr} - \pi^2 EI/(0.7l)^2}{\pi^2 EI/(0.7l)^2} \right| =$$

$$\left| \frac{20.08 - 20.14}{20.14} \right| = 0.003\,08 = 0.308\%.$$

与欧拉公式得到的误差为 0. 308%,验证了算法的有效性.

3.2　变截面梁求解算例

图 3 为长度 $l = 1$ m 的变截面杆件,弯曲刚度为 $EI(x)$,变截面高度为 h,小头截面高度为 $a = 20$ mm,大头截面高度为 $b = 40$ mm,截面宽度为 $c = 20$ mm. 求解该变截面杆件的临界荷载.

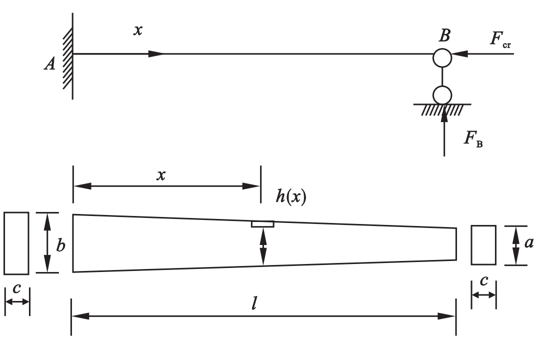

图 3　一端固定一端铰支受压杆件变截面示意图

Fig. 3　Sketch map of variable cross-section compression bar with one end fixed and one end hinged

变截面梁临界荷载求解. 由于杆件各个截面的惯性矩随截面尺寸的变化而变化,即关系式为

$$h(x) = \frac{b + (a - b)x}{l}, I(x) = \frac{1}{12} ch^3(x),$$

$$b(x) = \frac{EI(x)}{EI_0} = \frac{h^3(x)}{b^3}. \tag{13}$$

取段数 $m = 20$,此时设计变量为 $n = m + 1 = 21$,取初值:$z_i = \mathrm{rand}()$, $i = 1, 2, \cdots, 21$,其中,$z_1 \sim z_{19}$ 表示挠度,z_{20} 表示无量纲临界荷载 f_r,z_{21} 表示无量纲多余约束 f_b,多维 Powell 无约束精度 $e = 10^{-7}$,黄金分割精度 $e_1 = 10^{-4}$. 优化程序通过 3 273 轮,共 425 490 次优化计算,达到程序设定精度. 设计变量优化结果如表 2 所示.

因此,对应的无量纲临界载荷和无量纲多余约束为

$$f_r = z_{20} = 0.018\,3,\ f_b = z_{21} = -0.023\,9.$$

则临界荷载与多余约束分别为

$$F_{cr} = 0.018\,3 \frac{EI_0}{\Delta x^2} = 0.018\,3 \times 400 \times \frac{EI_0}{l^2} =$$

$$0.018\,3 \times 400 \times 200\,000 \times \frac{1}{12} \times \frac{cb^3}{l} =$$

$$0.018\,3 \times 400 \times 200\,000 \times \frac{1}{12} \times \frac{20 \times 40^3}{1\,000^2} =$$

$$156\,160\ \mathrm{N} = 156.160\ \mathrm{kN},$$

$$F_B=\frac{fbEI}{\Delta x^2}=F_B=-0.023\ 859\frac{EI_0}{\Delta x^2}=-0.023\ 859\times400\times\frac{EI_0}{l^2}=-0.023\ 859\times400\times200\ 000\times\frac{1}{12}\times\frac{cb^3}{l}=-0.023\ 859\times400\times200\ 000\times\frac{1}{12}\times\frac{20\times40^3}{1\ 000^2}=-203\ 596.8\ \text{N}=-203.597\ \text{kN}.$$

即所求变截面梁的临界荷载为156.160 kN,右端多余约束力为203.597 kN.

表2 $n=21$ 设计变量的优化计算结果

Table 2 Optimization result of design variables when $n=21$

动态变量	计算结果	动态变量	计算结果	动态变量	计算结果
z_1	0.011 9	z_8	0.705 0	z_{15}	1.267 7
z_2	0.048 1	z_9	0.855 2	z_{16}	1.150 9
z_3	0.108 3	z_{10}	1.000 0	z_{17}	0.958 7
z_4	0.191 6	z_{11}	1.129 6	z_{18}	0.693 0
z_5	0.296 2	z_{12}	1.233 1	z_{19}	0.365 5
z_6	0.419 5	z_{13}	1.298 6	z_{20}	0.018 3
z_7	0.557 5	z_{14}	1.314 0	z_{21}	-0.023 9

在设定精度条件下,根据所得结果绘制该变截面梁在临界荷载作用下的挠度曲线(见图4).

采用加密离散节点方法对超静定变截面压杆临界荷载求解结果评定.取段数 $m=40$,加密点增加一倍,此时设计变量为 $n=m+1=41$.取初值:$z_i=\text{rand}(\ )$,$i=1,2,\cdots,21$.其中 $z_1\sim z_{39}$ 表示挠度,z_{40} 表示无量纲临界荷载 f_r,z_{41} 表示无量纲多余约束力 f_b,多维 Powell 无约束精度 $e=10^{-7}$,黄金分割精度 $e_1=10^{-4}$.优化程序经5 213轮优化计算,共213 733次,达到程序设定精度.将设计变量优化结果如表3所示.

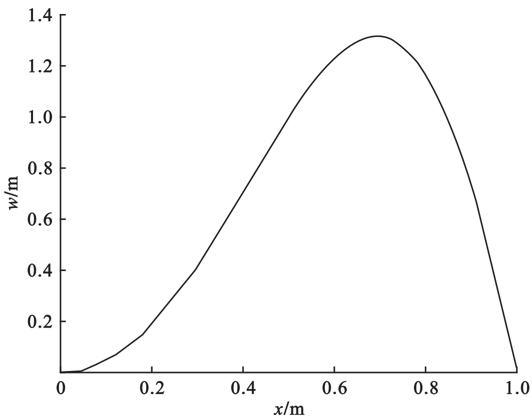

图4 杆件在临界荷载作用下的挠度曲线

Fig. 4 The deflection curve of bar under critical load

表3 $n=41$ 设计变量的优化计算结果

Table 3 Optimization result of design variables $z_1\sim z_{41}$ when n = 41

动态变量	计算结果	动态变量	计算结果	动态变量	计算结果
z_1	0.002 9	z_{15}	0.613 3	z_{29}	1.376 6
z_2	0.011 5	z_{16}	0.688 9	z_{30}	1.350 6
z_3	0.025 8	z_{17}	0.766 4	z_{31}	1.304 9
z_4	0.045 9	z_{18}	0.844 8	z_{32}	1.238 4
z_5	0.071 6	z_{19}	0.923 1	z_{33}	1.150 4
z_6	0.103 1	z_{20}	1.000 0	z_{34}	1.040 5
z_7	0.140 1	z_{21}	1.074 3	z_{35}	0.909 2
z_8	0.182 6	z_{22}	1.144 5	z_{36}	0.757 4
z_9	0.230 5	z_{23}	1.209 0	z_{37}	0.587 1
z_{10}	0.283 5	z_{24}	1.266 2	z_{38}	0.401 3
z_{11}	0.341 4	z_{25}	1.314 2	z_{39}	0.203 8
z_{12}	0.403 8	z_{26}	1.351 2	z_{40}	0.004 5
z_{13}	0.470 2	z_{27}	1.375 3	z_{41}	-0.006 0
z_{14}	0.540 2	z_{28}	1.384 3		

因此,对应的无量纲临界载荷和无量纲多余约束为

$f_r = z_{40} = 0.0045, f_b = z_{41} = -0.0060.$

由于设定相对误差值皆为 $\varepsilon_r = 5\%$.

则:
$$\left|\frac{F_r^{(k+1)} - F_r^{(k)}}{F_r^{(k)}}\right| = \left|\frac{0.0045 \times 1600 - 0.0183 \times 400}{0.0183 \times 400}\right| = 2.3\% < \varepsilon_r.$$

$$\sum_{i=1}^{n} \frac{1}{n}\left|\frac{w_i^{(k+1)} - w_i^{(k)}}{w_i^{(k)}}\right| = 4.25\% < \varepsilon_r.$$

表明加密节点[19-20]后的临界荷载误差和位移误差均在 5% 误差限内. 表明为所需临界载荷.

4　结　论

(1)基于差分和优化基本原理,提出一端固定,一端链杆约束的一类超静定变截面梁临界载荷的优化算法原理,编程分析计算了常截面和变截面压杆临界载荷求解算例.

(2)通过对比和误差分析,算法能够有效解决工程中变截面超静定梁结构临界载荷的计算问题,为工程设计与分析提供所需支持.

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